

# ON A DIOPHANTINE INEQUALITY INVOLVING A PRIME AND AN ALMOST-PRIME

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ABSTRACT. We prove that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_r| < p^{-\tau},$$

where  $r = 3$ ,  $\tau = \frac{1}{118}$ , and  $\lambda_0$  is an arbitrary real number and  $\lambda_1, \lambda_2 \in \mathbb{R}$  with  $\lambda_2 \neq 0$  and  $0 > \frac{\lambda_1}{\lambda_2}$  not in  $\mathbb{Q}$ . This improves a result by Harman. Moreover, we show that one can require the prime  $p$  to be of the form  $\lfloor n^c \rfloor$  for some positive integer  $n$ , i.e.  $p$  is a Piatetski-Shapiro prime, with  $r = 13$  and  $\tau = \rho(c)$ , a constant explicitly determined by  $c$  supported in  $(1, 1 + \frac{1}{149}]$ .

## CONTENTS

|                                       |    |
|---------------------------------------|----|
| 1. Introduction                       | 1  |
| 2. Notation and outline of the method | 2  |
| 2.1. Notation                         | 2  |
| 2.2. The weighted sieve               | 3  |
| 3. Some auxiliary lemmas              | 5  |
| 4. Estimates for exponential sums I   | 6  |
| 5. Sieve estimates                    | 9  |
| 6. Proof of Theorem 1                 | 18 |
| 7. Estimates for exponential sums II  | 20 |
| 8. Proof of Theorem 2                 | 22 |
| References                            | 23 |

## 1. INTRODUCTION

In Diophantine Approximation, a classical theorem of Kronecker ([4], Theorem 440) indicates that there are infinitely many solutions in positive integers  $n_1, n_2$  of

$$|\lambda_0 + \lambda_1 n_1 + \lambda_2 n_2| < 3 \left( \max \left\{ \frac{n_1}{\lambda_2}, \frac{n_2}{\lambda_1} \right\} \right)^{-1},$$

where  $\frac{\lambda_1}{\lambda_2}$  is irrational and  $\lambda_0$  is an arbitrary real number.

The case where  $n_1$  and  $n_2$  are both primes is of great interest and remains open to date ([12], [13]). The first approximation in this direction has been given by Vaughan [14] who proved that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_4| < p^{-1/600000},$$

where and henceforth in this paper the letter  $p$  denotes a prime and  $P_r$  a number with at most  $r$  prime factors. Harman [6] proved that there are infinitely many solutions of

$$(1) \quad |\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\tau},$$

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with  $\tau = \frac{1}{300}$ .

In this paper, we will improve Harman's result by showing that in (1) one can actually take  $\tau = \frac{1}{118}$ . One of the main results of this paper will be the following.

**Theorem 1.** *For  $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$  with  $\frac{\lambda_1}{\lambda_2}$  both negative and irrational, there are infinitely many solutions of*

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{118}}.$$

Moreover, recall that in [7] Heath-Brown proved Pjatecki-Šapīro prime number theorem, i.e.

$$\pi_c(x) := \sum_{\substack{n \leq x \\ \lfloor n^c \rfloor \text{ is a prime}}} 1 = c^{-1} Li(x) + O\left(xe^{-\delta\sqrt{\log x}}\right),$$

where  $c$  is a real number satisfying that  $1 < c < \frac{755}{662} = 1.1404\dots$ , and  $\delta = \delta(c) > 0$ . Thus we can naturally ask, what will happen if we replace the prime number theorem in the main term by Pjatecki-Šapīro prime number theorem? Can we require the prime  $p$  in Theorem 1 to be a Pjatecki-Šapīro prime?

The answer is positive, although at cost of increasing the number of factors of the corresponding almost-prime, and we will give a concrete describe about it as follows.

**Theorem 2.** *For  $c \in (1, 1 + \frac{1}{149}]$ ,  $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$  with  $\frac{\lambda_1}{\lambda_2}$  both negative and irrational, there are infinitely many solutions of*

$$|\lambda_0 + \lambda_1 \tilde{p} + \lambda_2 P_{13}| < \tilde{p}^{-\rho(c)},$$

where  $\tilde{p}$  is a prime of the form  $\lfloor n^c \rfloor$  for some positive integer  $n$  and

$$\rho(c) := \frac{1 + 9(c^{-1} - 1)}{12} - \frac{c}{13 - 0.144}.$$

*Remark.* We can take  $\rho(c) = \frac{1}{180}$ , when  $c = 1 + 2 \times 10^{-10}$ .

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## 2. NOTATION AND OUTLINE OF THE METHOD

**2.1. Notation.** We shall use  $\eta$  and  $\varepsilon$  for arbitrary small positive numbers (especially we require  $\varepsilon \leq \eta \leq 10^{-12}$ ); and sometimes they may be slightly different in context just for simplicity.

We write  $\lfloor x \rfloor$  for the largest integer not exceeding  $x$ . We write  $\|x\|$  for the distance from  $x$  to a nearest integer and  $[x]$  for the nearest integer to  $x$  when  $\|x\| \neq \frac{1}{2}$ . Clearly we may assume that  $\lambda_1 > 0$  and  $\lambda_2 = -1$ . Let  $\frac{a'}{q}$  be a convergence to the continued fraction for  $\lambda_1$  and assume  $q$  to be quite large in terms of  $\lambda_0, \lambda_1$  and  $\lambda_1^{-1}$ ; let  $X$  be a large number such that  $q \asymp X^{\frac{1}{3} + \rho + \eta}$ . Trivially, one can write  $\lambda_0 = \frac{b}{q} + \gamma$  with  $|\gamma| < \frac{1}{q}$ .

As in [6], we assume that  $q$  is so large that  $\min\{\frac{a'}{q}, \frac{q}{a'}\} > X^{-\frac{\rho}{4}}$  and  $a'X + b' < qX^{1+\frac{\eta}{4}}$ . In this paper,  $p, \tilde{p}, p_i, i = 1, 2, \dots$  represent primes;  $\sum^b$  indicates that the summation is only over square-free numbers. For convenience, we shall denote by

$$e(x) := \exp(2\pi i x), \quad \xi := X^{-\rho}, \quad \text{where } \rho \text{ is a positive number;}$$

$$P(z) := \prod_{p \leq z} p, \quad Y := \lfloor 3\xi^{-1} X^\eta \rfloor;$$

$$\pi_c(x) := \sum_{\substack{n \leq x \\ \lfloor n^c \rfloor \text{ is a prime}}} 1, \quad \pi(x) := \sum_{\substack{p \leq x \\ p \text{ is a prime}}} 1.$$

**2.2. The weighted sieve.** Essentially, to prove Theorem 1, if we use the same method as in [6] but with a parameterized weight to optimize the result, we will obtain that  $\tau = \frac{1}{147}$  is admissible as mentioned in Section 6. However, one can expect to obtain a better result by using Buchstab's sifting weights in [10] rather than Richert's weight  $w_p := 1 - \frac{u \log p}{\log X}$ , together with Selberg's trick, as in [8]. We will show in Theorem 14 that some terms in the resulting sums can be estimated more efficiently by using a 2-dimensional sieve, rather than using the linear sieve only. The 2-dimensional sieve helps us sieve primes in a much larger range, which will give a better result. Moreover, combining with Chen's idea, i.e., the so-called Switching Principle, as in [6], we can thus improve Harman's result. The last step is to work out the restrictions of those parameters both from main terms and error terms explicitly, and then figure out the optimal results from them, which can be done by *Mathematica* 9.

We will put the proof Theorem 2 in the last section, as it's somewhat similar to that of Theorem 1. For instance, the exponential sums appearing in the error terms can actually be divided into two parts roughly, one of which can actually be handled by results in Section 4. Nevertheless, we need a lemma to estimate the other part because it is an exponential sum of analytic type. All these will be done in Section 7.

Also, we will cover a slight gap of [6] in Section 4.

*Remark.* Selberg's trick can often help us slightly expand the range of sifting, e.g. see [9], where the sifting set is naturally multiplicative by the Chinese remainder theorem, and thus is easier to handle. However, the sifting set here has no multiplicative structure, so we have to use other tricks to conquer.

As it points out in [6] it suffices to show that the number of solutions of

$$\left| \frac{b'}{q} + \frac{pa'}{q} - P_3 \right| < \frac{X^{-\rho}}{2}$$

tends to infinity with  $X$ . Here  $p < X$ ,  $P_3 < \frac{a'X+b'}{q}$ . Hence, we will work with the set

$$\mathcal{A} := \left\{ \left\lfloor \frac{b' + pa'}{q} \right\rfloor : p \leq X, \left\| \frac{b' + pa'}{q} \right\| < \frac{\xi}{2} \right\}.$$

Here we list all notation used in the sieve method:

$$\begin{aligned} \mathcal{H}_r &:= \{n \in \mathcal{H} : r \mid n\}, \quad \text{for any finite set of positive integers } \mathcal{H}; \\ \mathfrak{N}(\beta) &:= \left\{ p_1 p_2 p_3 p_4 : X^\beta \leq p_1 < 2X^\beta, p_1 \leq p_2 \leq \left( \frac{a'X + b'}{qp_1} \right)^{\frac{1}{3}}, \right. \\ &\quad \left. p_2 \leq p_3 \leq \left( \frac{a'X + b'}{qp_1 p_2} \right)^{\frac{1}{2}}, X^{\frac{3}{4}} \leq p_4 \leq \frac{a'X + b'}{qp_1 p_2 p_3} \right\}; \\ \mathcal{A}(\beta)^* &:= \left\{ n : n \leq X, \left\| \frac{b' + na'}{q} \right\| < \frac{\xi}{2}, \left\lfloor \frac{b' + na'}{q} \right\rfloor \in \mathfrak{N}(\beta) \right\}; \\ \mathfrak{P}_r &:= \{n \in \mathbb{N} : n \text{ has at most } r \text{ prime divisors}\}; \end{aligned}$$

$$\begin{aligned}
R_d &:= \#\mathcal{A}_d - \frac{\pi(X)\xi}{d}; \quad \mathcal{S} := \sum_{n \in \mathcal{A} \cap \mathfrak{P}_3} 1; \\
\widetilde{w}_p &:= \begin{cases} cw_p, & \text{if } p = P_n \text{ or } p \geq x^{b/a}; \\ \min \left( cw_p, c - b - 1 + a \frac{\log P_n}{\log x} \right), & \text{otherwise,} \end{cases} \\
&\text{where } 1 \leq b \leq c \leq a = cu \text{ and } w_p := 1 - u \frac{\log p}{\log x}. \\
\mathcal{W}(\mathcal{A}, u, \lambda) &:= \sum_{\substack{s \in \mathcal{A} \\ (s, P(X^{\frac{1}{a}}))=1}}^b \left( 1 - \lambda \sum_{\substack{X^{\frac{1}{a}} \leq p \leq X^{\frac{c}{a}} \\ p|s}} \widetilde{w}_p \right) + \sum_{p \geq X^{\frac{1}{a}}} \sum_{\substack{h \in \mathcal{A} \\ p^2|h}} 1; \\
\mathcal{S}(\mathcal{A}(\beta)^*, z) &:= \sum_{\beta} \sum_{\substack{n \in \mathcal{A}(\beta)^* \\ (n, P(z))=1}} 1;
\end{aligned}$$

where  $0 < \frac{4}{a} \leq 4\beta \leq 1$ , both are undetermined parameters.

Define

$$\mathcal{J}(\lambda) := \mathcal{W}(\mathcal{A}, u, \lambda) - \lambda \mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}).$$

For simplicity, we shall denote by  $z := X^{\frac{1}{a}}$ ,  $y := X^{\frac{c}{a}}$ .

**Lemma 3.** Assume that  $b = 1$  or  $b > 1$  such that  $a \geq 3c + b + 1$ , then we have

$$(2) \quad \mathcal{S} \geq \mathcal{J}(\lambda) \quad \text{if } \lambda^{-1} < 5c - a.$$

*Proof.* Notice that

$$\mathcal{S} = \sum_{\substack{s \in \mathcal{A} \cap \mathfrak{P}_3 \\ (s, P(X^{\frac{1}{a}}))=1}}^b 1 + O(X^{1-\frac{a}{4}}),$$

thus we only need the following inequality:

$$\begin{aligned}
(3) \quad \sum_{\substack{s \in \mathcal{A} \setminus \mathfrak{P}_3 \\ (s, P(X^{\frac{1}{a}}))=1}}^b 1 &\leq \lambda \sum_{\substack{s \in \mathcal{A} \\ (s, P(X^{\frac{1}{a}}))=1}}^b \sum_{X^{\frac{a}{4}} \leq p \leq X^{\frac{1}{u}}} \widetilde{w}_p \\
&\quad + \lambda \sum_{\beta} \sum_{\substack{n \in \mathcal{A}(\beta)^* \\ (n, P(X^{\frac{1}{2}-\eta}))=1}} 1 + O(X^{1-\frac{a}{4}}),
\end{aligned}$$

with the assumption that  $0 < \rho < \frac{a}{4}$ . To this end, we divide it into two cases:

**Case 1:**  $s \in \mathcal{A} \setminus \mathfrak{P}_4$ , so that  $s$  has at least 5 prime factors. If  $s$  has a prime factor  $p$  which is larger than  $P_s$  and

$$\frac{\log p}{\log X} \leq \frac{b+1}{a} - \frac{\log P_s}{\log X},$$

then

$$\sum_{p|s} \widetilde{w}_p \geq c - a \frac{\log P_s}{\log X} + c - b - 1 + a \frac{\log P_s}{\log X} = 2c - b - 1 \geq 5c - a.$$

Otherwise, every prime divisor of  $s$  which is larger than  $P_s$  must satisfy

$$\frac{\log p}{\log X} \geq \frac{b+1}{a} - \frac{\log P_s}{\log X},$$

which means that

$$\widetilde{w}_p \geq c - a \frac{\log p}{\log X}, \quad \text{for all } p \mid s.$$

This provides that

$$\sum_{p \mid n} \widetilde{w}_p \geq c\omega(s) - a \frac{\log s}{\log X} \geq 5c - a.$$

thus we have (3) because of

$$\lambda^{-1} < 5c - a.$$

**Case 2:**  $s \in \mathcal{A} \cap \mathfrak{P}_4$  Similarly as above, we have  $\sum_{p \mid n} \widetilde{w}_p \geq 4c - a$ . So (3) comes from the assumption that  $\lambda^{-1} < 5c - a$ .

□

Therefore, we have

**Corollary 4.** For  $\lambda^{-1} < 5c - a$ , if

$$\mathcal{J}(\lambda) := \mathcal{W}(\mathcal{A}, u, \lambda) - \lambda \mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}) \gg \frac{\pi(X)\xi}{\log X},$$

then theorem 1 holds with  $\tau = \rho$ .

In the following sections, we will prove that  $\mathcal{J}(\lambda) \gg \frac{\pi(X)\xi}{\log X}$  and we can take  $\rho = \frac{1}{118}$ .

### 3. SOME AUXILIARY LEMMAS

**Lemma 5.** For any  $x \geq 2$ , we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right);$$

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right),$$

where  $c$  is an absolute constant.

*Remark.* These two estimates are usually called Mertens formulas.

**Lemma 6** ([11]). Let  $\delta_0 < \frac{1}{2}$  and  $\chi(t)$  be the characteristic function of interval  $(-\delta_0, \delta_0)$  extended to be periodic with period 1, then there exists  $A(t)$ ,  $B(t)$  such that

$$A(t) \leq \chi(t) \leq B(t)$$

where  $A(t)$ ,  $B(t)$  can be written as

$$A(t) := 2\delta_0 - (N+1)^{-1} + \sum_{1 \leq |n| \leq N} A_n e(nt),$$

$$B(t) := 2\delta_0 + (N+1)^{-1} + \sum_{1 \leq |n| \leq N} B_n e(nt),$$

with coefficients  $A_n, B_n$  satisfying  $\max\{|A_n|, |B_n|\} \ll \delta_0$ , for  $1 \leq |n| \leq N$ .

**Lemma 7.** Suppose that  $0 \leq \alpha < \beta \leq 1$  and  $\Delta > 0$  with  $2\Delta < \beta - \alpha$ , then there exists a smooth function  $\chi$  with the period 1 satisfying that:

- (1):  $\chi(x) = 1$  if  $\alpha + \Delta \leq \{x\} \leq \beta - \Delta$ ,  $\chi(x) = 0$  if  $\{x\} \leq \alpha$  or  $\{x\} \geq \beta$ , and  $\chi(x) \in [0, 1]$  otherwise.

(2):  $\chi(x) = \beta - \alpha + \sum_{1 \leq |h| \leq \Delta^{-1-\varepsilon}} c_h e(hx) + O(\Delta)$ , where

$$c_h \ll_{\varepsilon} \min\left\{\frac{1}{|h|}, \beta - \alpha - \Delta\right\}.$$

Moreover, the function  $g(x) := \sum_{1 \leq |h| \leq \Delta^{-1-\varepsilon}} c_h e(hx)$  is real.

*Proof.* Fix  $\varepsilon > 0$  small enough. Then by ([17] Lemma 12, Chapter 1) we have

$$\chi(x) = \beta - \alpha - \Delta + \sum_{|h| \geq 1} (a_j \cos 2\pi jx + b_j \sin 2\pi jx).$$

Take  $c_j = \frac{a_j - ib_j}{2}$  and  $c_{-j} = \frac{a_j + ib_j}{2}$  for any  $j \in \mathbb{N}_{\geq 1}$ , then by estimations from ([17]) on  $a_j$  and  $b_j$ , we have

$$c_h \ll_{\varepsilon} \min\left\{\frac{1}{|h|}, \beta - \alpha - \Delta, \frac{1}{\Delta^r |h|^{r+1}}\right\} \quad \text{for any arbitrary integer } r.$$

Take  $r$  large enough such that  $\frac{1}{r} \leq \varepsilon$  and  $H := \Delta^{-\frac{r+1}{r}}$ , then

$$\sum_{|h| \geq H} c_h e(hx) \ll \sum_{|h| \geq H} \frac{1}{\Delta^r |h|^{r+1}} \ll \frac{1}{\Delta^r |H|^r} \ll \Delta.$$

Obviously,  $g(x) = \sum_{1 \leq |h| \leq \Delta^{-1-\varepsilon}} (a_j \cos 2\pi jx + b_j \sin 2\pi jx)$  is a real function.  $\square$

Set

$$\mathcal{S}_{\tilde{w}}(\mathcal{A}) := \sum_{\substack{s \in \mathcal{A} \\ (s, P(X^{\frac{b+1}{2a}}))=1}} \sum_{\substack{X^{\frac{a}{4}} \leq p \leq X^{\frac{1}{u}} \\ p|s}} \widetilde{w_p},$$

then by a direct computation we have

**Lemma 8.**

$$\begin{aligned} \mathcal{S}_{\tilde{w}}(\mathcal{A}) &= \left(1 - \frac{b}{c}\right) \sum_{X^{\frac{1}{a}} \leq p < X^{\frac{b}{a}}} \mathcal{S}\left(\mathcal{A}_p, X^{\frac{1}{a}}\right) + u \int_{\frac{1}{a}}^{\frac{b+1}{2a}} \left( \sum_{X^s \leq p \leq X^{\frac{b+1}{a}-s}} \mathcal{S}(\mathcal{A}_p, X^s) ds \right) \\ &\quad + \sum_{X^{\frac{1}{a}} \leq p < X^{\frac{b+1}{2a}}} \left( \frac{b+1}{c} - \frac{2u \log p}{\log X} \right) \mathcal{S}(\mathcal{A}_p, p) + \sum_{X^{\frac{b}{a}} \leq p < X^{\frac{1}{u}}} w_p \mathcal{S}\left(\mathcal{A}_p, X^{\frac{1}{a}}\right). \end{aligned}$$

#### 4. ESTIMATES FOR EXPONENTIAL SUMS I

Our main goal in this section is to prove that

$$(4) \quad \sum_{d \leq X^{\alpha}} \frac{\xi}{d} \max_{N \leq X} \sum_{l=1}^{dY} \left| \sum_{n \leq N} \Lambda(n) e\left(\frac{anl}{dq}\right) \right| \ll \xi \pi(X) X^{-\eta}$$

with  $\alpha$  as large as possible.

However, the lemmas in [6] can only give the result without taking *max* between the two sums. We should point out that with some slight modifications of the proof in [6] we will be able to prove (4).

This is a generalization of [6], Lemma 3:

**Lemma 9.** Suppose  $X, M \geq 1$ ,  $\delta > 0$ ,  $\mathcal{M}$  a set of  $\leq T$  integer points  $(l, m)$  with  $M \leq m < 2M$ ,  $\lambda_{lm}$  real numbers for  $(l, m) \in \mathcal{M}$ , and  $\{a_n\}$  a sequence of complex numbers, then

$$\sum_{(l, m) \in \mathcal{M}} \max_{N \leq X} \left| \sum_{mn \leq N} a_n e(\lambda_{lm} n) \right|^2 \ll D_{\delta} \log^3(2TX) \left( \frac{X}{M} + \delta^{-1} \right) \sum_{n \leq X/M} |a_n|^2,$$

where

$$D_\delta = \max_{(l,m) \in \mathcal{M}} \# \{(l', m') \in \mathcal{M} : \|\lambda_{lm} - \lambda_{l'm'}\| < \delta\}.$$

*Proof.* Define

$$\delta(\beta) := \begin{cases} 1, & \text{if } 0 \leq \beta \leq \gamma, \\ 0, & \text{otherwise,} \end{cases}$$

which is a truncation function. Then we have

$$\delta(\beta) = \int_{-A}^A e^{i\beta t} \frac{\sin \gamma t}{\pi t} dt + O\left(\frac{1}{A|\gamma - \beta|}\right)$$

as in the proof of Lemma 2 of [16]. Here we take  $A = 2TX$ ,  $\gamma_{lm} = \log(N_{lm} + \frac{1}{2})$ , for  $(l, m) \in \mathcal{M}$ . where

$$N_{lm} = \max \left\{ n_0 \in X : \max_{N \leq X} \left| \sum_{mn \leq N} a_n e(\lambda_{lm} n) \right|^2 = \left| \sum_{mn \leq n_0} a_n e(\lambda_{lm} n) \right|^2 \right\}.$$

Then we have

$$\begin{aligned} & \sum_{(l,m) \in \mathcal{M}} \max_{N \leq X} \left| \sum_{mn \leq N} a_n e(\lambda_{lm} n) \right|^2 = \sum_{(l,m) \in \mathcal{M}} \left| \sum_{mn \leq N_{lm}} a_n e(\lambda_{lm} n) \right|^2 \\ & \leq \sum_{(l,m) \in \mathcal{M}} \left( \int_{-A}^A \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right| \cdot \left| \frac{\sin \gamma_{lm} t}{\pi t} \right| dt \right)^2 \\ & \quad + O \left( \sum_{(l,m) \in \mathcal{M}} \left( \sum_n |a_n| \frac{1}{A \log \frac{N_{lm} + 1/2}{mn}} \right)^2 \right) \\ & \ll \sum_{(l,m) \in \mathcal{M}} \int_{-A}^A \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \cdot \min\{\gamma_{lm}, \frac{1}{|t|}\} dt \cdot \log A \\ & \quad + O \left( \sum_{(l,m) \in \mathcal{M}} \left( \sum_n |a_n| \frac{1}{A \log \frac{N_{lm} + 1/2}{N_{lm}}} \right)^2 \right) \\ & \ll \log A \cdot \int_{-A}^A \left( \sum_{(l,m) \in \mathcal{M}} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \right) \cdot \min\{\log X, \frac{1}{|t|}\} dt \\ & \quad + O \left( \sum_{(l,m) \in \mathcal{M}} \left( \sum_n |a_n| \frac{1}{A \log \frac{X+1/2}{X}} \right)^2 \right) \\ & \ll \log A \cdot \left( \sum_{(l,m) \in \mathcal{M}} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \right) \cdot \int_{-A}^A \min\{\log X, \frac{1}{|t|}\} dt \\ & \quad + O \left( \sum_{(l,m) \in \mathcal{M}} \left( \sum_n |a_n| \frac{1}{A \log \frac{X+1/2}{X}} \right)^2 \right) \\ & \ll D_\delta \log^2(2TX) \left( \frac{X}{M} + \delta^{-1} \right) \left( \int_0^1 \log X dt + \int_1^A \frac{1}{t} dt \right) \sum_n |a_n|^2 \\ & \ll D_\delta \log^3(2TX) \left( \frac{X}{M} + \delta^{-1} \right) \sum_n |a_n|^2, \end{aligned}$$

where the last step comes from [6], Lemma 3.  $\square$

This is a generalization of [6], Lemma 5:

**Lemma 10.** *Suppose  $\varepsilon > 0$ ,  $X > R$ ,  $J, M \geq 1$ ,  $1 < q \leq X^{\frac{3}{4}}$ ,  $\log |a| \ll \log X$ ,  $(a, q) = 1$ , then*

$$\sum_{r \sim R} \max_{N \leq X} \sum_{j \sim J} \sum_{m \sim M} \left| \sum_{mn \leq N} e \left( \frac{ajmn}{rq} \right) \right| \ll X^\varepsilon \left( \frac{JX}{q} + RJM + qR^2 \right).$$

*Proof.* By lemma 3 of [15] we obtain

$$\begin{aligned} \sum_{r \sim R} \max_{N \leq X} \sum_{j \sim J} \sum_{m \sim M} \left| \sum_{mn \leq N} e \left( \frac{ajmn}{rq} \right) \right| \\ \ll \log X (JM)^{\frac{\varepsilon}{3}} \sum_{r \sim R} \left( \frac{JX \cdot (r, a)}{rq} + JM + qR \right). \end{aligned}$$

Hence, it follows from the same estimates in lemma 5 of [6].  $\square$

This is a generalization of [6], Lemma 7:

**Lemma 11.** *Suppose that  $\varepsilon > 0$ ,  $X \geq R$ ,  $L, M \geq 1$ ,  $1 < q \leq X$ ,  $(a, q) = 1$  and  $a \asymp q$ ,  $\max \left\{ \frac{LM}{qR}, \frac{qM}{X} \right\} < 1$ ,  $a_n, b_m \ll X^\varepsilon$ . Then*

$$\begin{aligned} \sum_{r \sim R} \max_{N \leq X} \sum_{l \sim L} \left| \sum_{m \sim M} b_m \sum_{mn \leq N} a_n e \left( \frac{lmna}{qr} \right) \right| \\ \ll X^{1+3\varepsilon} R \left( L + \frac{R}{M} \right) \left( \frac{M}{X} + \frac{1}{MRL + R^2} \right)^{\frac{1}{2}}. \end{aligned}$$

*Proof.* The proof is essentially the same as that of lemma 7 of [6], with lemma 3 of [6] replaced by lemma 9 above.  $\square$

This is a generalization of [6], Lemma 8:

**Lemma 12.** *Suppose that  $X, R, L \geq 1$ ,  $a \asymp q$ ,  $(a, q) = 1$ ,  $\varepsilon > 0$  and  $\frac{TX^{\frac{1}{3}}}{R} < q < X^{\frac{2}{3}}$ , where  $T = \max\{L, R\}$ . Then we have*

$$\sum_{r \sim R} \max_{N \leq X} \sum_{l \sim L} \left| \sum_{n \leq N} \Lambda(n) e \left( \frac{anl}{rq} \right) \right| \ll X^\varepsilon \left( X^{\frac{2}{3}} TR + X^{\frac{11}{12}} (TR)^{\frac{1}{2}} \right).$$

*Proof.* Using Vaughan's identity we split the inner sum above into  $\ll \log N$  sums of the form

$$\sum_{m \sim M} \sum_{mn \leq N} a_n b_m e \left( \frac{nalm}{dq} \right),$$

with either

(I)  $a_n = 1$  or  $\log n$ ,  $M < X^{\frac{2}{3}}$ ,  $b_m \ll X^\varepsilon$ , or

(II)  $a_n, b_m \ll X^\varepsilon$ ,  $X^{\frac{1}{6}} < M < X^{\frac{1}{3}}$ .

Sums of type (I) can be handled by lemma 10 and sums of type (II) by lemma 11 and the estimate above follows.  $\square$

**Corollary 13.** *We have*

$$\sum_{d \leq X^\alpha} \frac{\xi}{d} \max_{N \leq X} \sum_{l=1}^{dY} \left| \sum_{n \leq N} \Lambda(n) e \left( \frac{anl}{dq} \right) \right| \ll \xi \pi(X) X^{-\eta}.$$



## 5. SIEVE ESTIMATES

Let  $f_1$ ,  $F_1$  and  $F_2$  be the limit functions occurred in Beta-Sieve, which are given by the following definition:

$$\begin{aligned} f_1(s) &:= A_1 s^{-1} \log(s-1) && \text{for } 2 \leq s \leq 4; \\ f_1(s) &:= A_1 s^{-1} \left( \log(s-1) + \int_3^{s-1} \frac{du}{u} \int_2^{u-1} \frac{\log(v-1)}{v} dv \right) && \text{for } 4 \leq s \leq 6; \\ F_1(s) &:= A_1 s^{-1} && \text{for } s \leq 3; \\ F_1(s) &:= A_1 s^{-1} \left( 1 + \int_2^{s-1} \frac{\log(v-1)}{v} dv \right) && \text{for } 3 \leq s \leq 5; \\ F_2(s) &:= A_2 s^{-2} && \text{for } s \leq \beta_2 + 1, \end{aligned}$$

where  $A_1 = 2e^\gamma$ ,  $\beta_2 = 4.8333\cdots$ ,  $A_2 = 43.496\cdots$  are defined in [3], Chapter 11. We can, with a patient calculation, show that for  $s \in [\beta_2 + 1, \beta_2 + 2)$ , we have

$$\begin{aligned} F_2(s) &= s^{-2} \left( \frac{2A_2 \log \beta_2}{s-1} + C_0 + 2A_2 \log^2(s-1) + 4A_2 \log(s-1) \right) \\ &\quad - \frac{4A_2 (1 + s \log(s-1))}{s^2(s-1)}, \end{aligned}$$

where  $C_0$  is determined by  $F_2(\beta_2 + 1) = \frac{A_2}{(\beta_2 + 1)^2}$ . As shown in Lemma 15 below, the level of distribution of  $\mathcal{A}$  can be taken as  $\theta_1 = \frac{1}{3} - \rho - \varepsilon$ . Henceforth, we take  $a = \frac{\vartheta}{\theta_1}$  and optimize  $\vartheta$  to get a better upper bound of  $\rho$ . Take  $z = X^{\frac{1}{a}}$  and  $y = X^{\frac{c}{a}}$  from now on.

*Remark.* The limit functions  $f_1$  and  $F_2$  are actually defined by systems of differential equations piecewise respectively.  $f_1$  is increasing rapidly and very close to its limit 1 when  $s \geq 6$ . While  $F_2$  is decreasing with limit 1. We should point out that in our situation, it turns out that  $6\theta - c > \beta_2 + 1$  since we require that  $b \geq 3$ , which leads  $c$  to be relatively small. Thus the above expression of  $F_2$  is invalid. We will discuss this matter in the next section.

Denote by  $A_3 := \frac{A_2}{2e^{2\gamma}} \approx 6.85577$ , which will be used in the following section.

In this section, we will prove the following theorem, which improves [6], Lemma 1:

**Theorem 14.** *Let notations be defined as before and assume that  $b = 1$  or  $b > 1$  such that  $a \geq 3c + b + 1$ , then for any  $\delta \in [\frac{b}{\vartheta}, \frac{c}{\vartheta}]$  we have,*

$$\mathcal{J}(\lambda) \geq \frac{ae^{-\gamma}(1 + o(1))\lambda\xi\pi(X)}{\log X} \mathcal{H}_\delta(\vartheta, b, c),$$

where

$$(5) \quad \mathcal{H}_\delta(\vartheta, b, c) = 2e^\gamma (A_\delta(\vartheta)b + B_\delta(\vartheta)c + D_\delta(\vartheta) + \mathfrak{F}_\delta(\vartheta, c)),$$

with

$$\begin{aligned} A_\delta(\vartheta) &= -e^{-\gamma} f_1(\vartheta) + \frac{1}{2e^\gamma} \int_{\frac{1}{\vartheta}}^\delta F_1(\vartheta(1-s)) \frac{ds}{s} + \frac{1}{\vartheta} \log \frac{1-\delta}{\delta}; \\ B_\delta(\vartheta) &= e^{-\gamma} f_1(\vartheta) - \frac{1}{2e^\gamma} \int_{\frac{1}{\vartheta}}^\delta F_1(\vartheta(1-s)) \frac{ds}{s} - \frac{2}{a} \mathfrak{I}(\rho); \\ D_\delta(\vartheta) &= \frac{1}{2e^\gamma} \mathcal{H}(\vartheta, \vartheta\theta, \vartheta\theta) - \delta \log \frac{1-\delta}{\delta} + \frac{2\vartheta\theta}{a} \mathfrak{I}(\rho); \\ \mathfrak{F}_\delta(\vartheta, c) &= -ae^{-\gamma} \left( \int_\delta^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_2(a\theta_2 - \vartheta s) ds \right). \end{aligned}$$

To this end, we need the following lemmas.

**Lemma 15.** *We have*

$$\mathcal{S}(\mathcal{A}, z) \geq \xi \pi(X) V(z) (f_1(6) + o(1)),$$

where  $V(z) = \frac{e^{-\gamma}}{\log z} (1 + o(1))$ , and  $z := X^{\frac{1}{a}}$  as mentioned before.

*Proof.* Take  $M \asymp \frac{dX^\eta}{\xi}$  in Lemma 6 then we have

$$\begin{aligned} \#\mathcal{A}_d &= \sum_{\substack{p \leq X \\ d \mid \lceil \frac{ap+b}{q} \rceil \\ \|\frac{ap+b}{q}\| < \frac{\xi}{2}}} 1 = \sum_{\substack{p \leq X \\ \|\frac{ap+b}{dq}\| < \frac{\xi}{2d}}} 1 = \sum_{p \leq X} \chi\left(\frac{ap+b}{dq}\right) \\ &= \frac{\pi(x)\xi}{d} + E(\mathcal{A}_d) + O\left(\frac{\xi \pi(X) X^{-\eta}}{d}\right), \end{aligned}$$

where

$$\sum_{p \leq X} \sum_{1 \leq |l| \leq M} a_l e\left(\frac{(ap+b)l}{qd}\right) \leq E(\mathcal{A}_d) \leq \sum_{p \leq X} \sum_{1 \leq |l| \leq M} b_l e\left(\frac{(ap+b)l}{qd}\right)$$

with  $|a_l| + |b_l| \ll \frac{\xi}{d}$ ,  $\forall 1 \leq |l| \leq M$ .

Therefore, by partial summation we have

$$\begin{aligned} E(\mathcal{A}_d) &\ll \max_{N \leq X} \frac{1}{\log X} \sum_{1 \leq |l| \leq M} (|a_l| + |b_l|) \left| \sum_{n \leq N} \Lambda(n) e\left(\frac{anl}{qd}\right) \right| \\ &\ll \max_{N \leq X} \frac{\xi}{d} \sum_{l=1}^{dY} \left| \sum_{n \leq N} \Lambda(n) e\left(\frac{anl}{qd}\right) \right|. \end{aligned}$$

Hence the density function of sequence  $\mathcal{A}$  is  $g_1(d) = \frac{1}{d}$ ; and thus, by Jurkat-Richert's theorem, we obtain

$$\begin{aligned} \mathcal{S}(\mathcal{A}, z) &\geq \xi \pi(X) V(z) \left( f_1(4) + O\left((\log X)^{-\frac{1}{6}}\right) \right) \\ &\quad + O\left( \xi \pi(X) X^{-\eta} + \sum_{d \leq X^\alpha} \frac{\xi}{d} \max_{N \leq X} \sum_{i=1}^{dY} \left| \sum_{n \leq N} \Lambda(n) e\left(\frac{anl}{dq}\right) \right| \right). \end{aligned}$$

Then this lemma comes from corollary 13 since  $f_1(6) > 0$ .  $\square$

**Lemma 16.** *If  $0 < \frac{1}{a} < \delta' < \frac{\varepsilon}{a} \leq \theta_1$ , let  $w = X^{\delta'}$ , then*

$$\sum_{z \leq p < w} w_p \mathcal{S}(\mathcal{A}_p, z) \leq \xi \pi(X) V(z) \left( \int_{\frac{1}{a}}^{\delta'} \left( \frac{1}{s} - u \right) F_1 \left( \frac{4(\alpha - s)}{\alpha} \right) ds + o(1) \right).$$

*Proof.* Corollary 13 shows that the level of distribution of  $\mathcal{A}$  is  $X^{\theta_1}$ . Hence by Jurkat-Richert's theorem, we have

$$\mathcal{S}(\mathcal{A}_p, z) \leq \frac{\xi \pi(X) V(z)}{p} \left( F_1(s_p) + O \left( \left( \log \frac{X^\alpha}{p} \right)^{-\frac{1}{\theta_1}} \right) \right) + O \left( \sum_{d \leq X^{\theta_1}/p} |R_{pd}| \right),$$

where

$$s_p = \frac{\log \frac{X^{\theta_1}}{p}}{\log z}, \text{ and } R_{pd} \ll \frac{\xi}{pd} \max_{N \leq X} \sum_{l=1}^{pdY} \left| \sum_{n \leq N} \Lambda(n) e \left( \frac{anl}{pdq} \right) \right|.$$

Since

$$\begin{aligned} \sum_{z \leq p < w} w_p \sum_{d \leq X^{\theta_1}/p} |R_{pd}| &\ll \sum_{z \leq p < w} \sum_{d \leq X^{\theta_1}/p} |R_{pd}| \\ &\ll \sum_{d \leq X^{\theta_1}} |R_d| \sum_{\substack{p|d \\ z \leq p < w}} 1 \\ &\ll X^{\frac{\eta}{2}} \sum_{d \leq X^{\theta_1}} |R_d| \\ &\ll X^{\frac{\eta}{2}} \sum_{d \leq X^{\theta_1}} \frac{\xi}{d} \max_{N \leq X} \sum_{l=1}^{dY} \left| \sum_{n \leq N} \Lambda(n) e \left( \frac{anl}{dq} \right) \right| \ll X^{1-\frac{\eta}{2}} \xi, \end{aligned}$$

we obtain

$$\sum_{z \leq p < w} w_p \mathcal{S}(\mathcal{A}_p, z) \leq \xi \pi(X) V(z) \left( \sum_{z \leq p < w} \frac{w_p}{p} F_1(s_p) + o(1) \right).$$

By Mertens formula we have

$$\sum_{t' \leq p < t} \frac{1}{p} \left( 1 - \frac{\log p}{\log y} \right) = \log \frac{\log t}{\log t'} - \frac{\log t - \log t'}{\log y} + R(t', t),$$

where

$$R(t', t) \ll \frac{1}{t'}, \quad \text{for any } t' < t.$$

Notice that  $F_1$  is bounded and decreasing, so we obtain that

$$\begin{aligned} \sum_{z \leq p < w} \frac{w_p}{p} F_1(s_p) &= \int_z^w F_1 \left( \frac{\log X^{\theta_1}/t}{\log z} \right) d \sum_{z \leq p < t} \frac{1}{p} \left( 1 - \frac{\log p}{\log y} \right) \\ &= \int_z^w F_1 \left( \frac{\log X^{\theta_1}/t}{\log z} \right) d \left( \log \frac{\log t}{\log z} - \frac{\log t - \log z}{\log y} + R(z, t) \right) \\ &= \int_z^w \frac{1}{t} \left( \frac{1}{\log t} - \frac{1}{\log y} \right) F_1 \left( \frac{\log X^{\theta_1}/t}{\log z} \right) dt + R(z, t) F_1 \left( \frac{\log X^{\theta_1}/t}{\log z} \right) \Big|_z^w \\ &\quad + O \left( \int_z^w \frac{1}{z} dF_1 \left( \frac{\log X^{\theta_1}/t}{\log z} \right) \right) \\ &= \int_{\frac{1}{a}}^{\delta'} \left( \frac{1}{s} - u \right) F_1(a(\theta_1 - s)) ds + O\left(\frac{1}{z}\right). \end{aligned}$$

Therefore, we have

$$\sum_{z \leq p < w} w_p \mathcal{S}(\mathcal{A}_p, z) \leq \xi \pi(X) V(z) \left( \int_{\frac{1}{a}}^{\delta'} \left( \frac{1}{s} - u \right) F_1(a(\theta_1 - s)) ds + o(1) \right).$$

This completes the proof.  $\square$

Define

$$\tilde{\mathcal{A}} := \left\{ n \left\lceil \frac{an+b}{q} \right\rceil : n \in [z, X], p \mid \left\lceil \frac{an+b}{q} \right\rceil, \left\| \frac{an+b}{q} \right\| < \frac{1}{2}\xi \right\},$$

then we have the following auxiliary lemma.

**Lemma 17.** *For  $d \mid P(z)$  and  $p \geq z$ , we have*

$$\#\tilde{\mathcal{A}}_d = \frac{X\xi}{p} g_2(d) + E(X; p, d),$$

where

$$g_2(d) := \prod_{p \mid d} \left( \frac{2}{p} - \frac{1}{p^2} \right),$$

$$E(X; p, d) \ll q\xi\tau(d) + \frac{X}{pdq} \sum_{d=d_1d_2} (q, d_1)(a, pd_2).$$

*Proof.* Define

$$J := \{j : |j+b| \leq \frac{1}{2}q\xi\}.$$

$$\begin{aligned} \#\tilde{\mathcal{A}}_d &= \sum_{|j| \leq \frac{1}{2}q\xi} \sum_{\substack{n \in [z, X] \\ an+b \equiv j \pmod{pq} \\ n(an+b) \equiv jn \pmod{dq}}} 1 = \sum_{j \in J} \sum_{\substack{n \in [z, X] \\ an \equiv j \pmod{pq} \\ an^2 \equiv jn \pmod{dq}}} 1 \\ &= \sum_{j \in J} \sum_{\substack{n \in [z, X] \\ an-j \equiv 0 \pmod{pq} \\ n(an-j) \equiv 0 \pmod{dq}}} 1 = \sum_{j \in J} \sum_{d=d_1d_2} \sum_{\substack{n \in [z, X] \\ an-j \equiv 0 \pmod{pq} \\ n(an-j) \equiv 0 \pmod{dq} \\ (n, d) = d_1}} 1 \\ &= \sum_{d=d_1d_2} \sum_{j \in J} \sum_{\substack{n \in [z/d_1, X/d_1] \\ ad_1n-j \equiv 0 \pmod{pq} \\ n(ad_1n-j) \equiv 0 \pmod{d_2q} \\ (n, d_2) = 1}} 1 = \sum_{d=d_1d_2} \sum_{j \in J} \sum_{\substack{n \in [z/d_1, X/d_1] \\ ad_1n-j \equiv 0 \pmod{pd_2q} \\ (n, d_2) = 1}} 1 \\ &= \sum_{d=d_1d_2} \sum_{\substack{j \in J \\ (ad_1, pd_2q) \mid j}} \left( \frac{\varphi(d_2)}{d_2} \cdot \frac{X-z}{pdq} (ad_1, pd_2q) + O(1) \right) \\ &= \sum_{d=d_1d_2} \left( \frac{q\xi}{(ad_1, pd_2q)} + O(1) \right) \left( \frac{\varphi(d_2)}{d_2} \cdot \frac{X-z}{pdq} (ad_1, pd_2q) + O(1) \right) \\ &= \sum_{d=d_1d_2} \frac{\varphi(d_2)}{d_2} \cdot \frac{(X-z)\xi}{pd} + O \left( \sum_{d=d_1d_2} \frac{q\xi}{(ad_1, pd_2q)} + \sum_{d=d_1d_2} \frac{X \cdot (ad_1, pd_2q)}{pdq} \right), \end{aligned}$$

and thus lemma follows by noting that  $(a, pd_2q)(d_1, pd_2q) \leq (q, d_1)(a, pd_2)$ .  $\square$

Hence  $\tilde{\mathcal{A}}$  has a density function  $g_2(d)$  with

$$\begin{aligned} V_2(z) &:= \prod_{p \leq z} (1 - g_2(p)) = \prod_{p \leq z} \left(1 - \frac{2}{p} + \frac{1}{p^2}\right) \\ &= \frac{e^{-2\gamma}}{\log^2 z} (1 + o(1)) \quad \text{by Mertens estimate.} \end{aligned}$$

We will use Beta-Sieve theory to  $\tilde{\mathcal{A}}$  to obtain an upper bound with a larger exponent of level of distribution. To this end, we shall compute its dimension as follows:

$$\sum_{p \leq v} g_2(p) \log p = 2 \sum_{p \leq v} \left( \frac{\log p}{p} - \frac{\log p}{2p^2} \right) = 2 \log v + O(1), \quad \text{for any } v \geq 2.$$

Therefore, the sieve dimension is 2. Denote by  $\theta_2$  the exponent of level of distribution of  $\tilde{\mathcal{A}}$ .

**Lemma 18.** *Assuming  $w \leq p \leq y$  and  $p$  is a prime number, where  $w = X^{\delta'}$ ,  $\frac{1}{a} \leq \delta \leq \frac{c}{a} \leq \theta_1$ , then we have*

$$\mathcal{S}(\mathcal{A}_p, z) \leq \frac{X\xi}{p} V_2(z) \left( F_2(s'_p) + O\left((\log X)^{-\frac{1}{6}}\right) \right) + \sum_{d \leq \frac{X^{\theta_2}}{p}} E(X; p, d),$$

where

$$s'_p := \frac{\log(X^{\theta_2}/p)}{\log z}.$$

*Proof.* We have

$$\begin{aligned} \mathcal{S}(\mathcal{A}_p, z) &= \sum_{\substack{n \in \mathcal{A}_p \\ (n, P(z))=1}} 1 \\ &= \#\{p' : z \leq p' \leq X, p \mid \left\lceil \frac{ap' + b}{q} \right\rceil, \left\| \frac{ap' + b}{q} \right\| < \frac{1}{2}\xi, \left( p' \left\lceil \frac{ap' + b}{q} \right\rceil, P(z) \right) = 1\} \\ &\quad + \#\{p' : p' < z, p \mid \left\lceil \frac{ap' + b}{q} \right\rceil, \left\| \frac{ap' + b}{q} \right\| < \frac{1}{2}\xi, \left( p' \left\lceil \frac{ap' + b}{q} \right\rceil, P(z) \right) = 1\} \\ &\leq \#\{n : z \leq n \leq X, p \mid \left\lceil \frac{an + b}{q} \right\rceil, \left\| \frac{an + b}{q} \right\| < \frac{1}{2}\xi, \left( n \left\lceil \frac{an + b}{q} \right\rceil, P(z) \right) = 1\} \\ &\quad + O(\xi\pi(z)) \\ &= \mathcal{S}(\tilde{\mathcal{A}}, z) + O(\xi\pi(z)), \end{aligned}$$

We now meet a sifting problem of dimension two. By Beta-Sieve theory we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}_p, z) &\leq \mathcal{S}(\tilde{\mathcal{A}}, z) + O(\xi\pi(z)) \\ &\leq \frac{X\xi}{p} V_2(z) \left( F_2(s'_p) + O\left(\left(\log \frac{X^{\theta_2}}{p}\right)^{-\frac{1}{6}}\right) \right) + \sum_{d \leq X^{\theta_2}/p} E(X; p, d) \\ &\quad + O(\xi\pi(z)) \\ &= \frac{X\xi}{p} V_2(z) \left( F_2(s'_p) + O\left((\log X)^{-\frac{1}{6}}\right) \right) + \sum_{d \leq X^{\theta_2}/p} E(X; p, d) \end{aligned}$$

and the last inequality holds because

$$\xi\pi(z) \ll X\xi V_2(z)(\log X)^{-\frac{1}{6}}.$$

This completes the proof.  $\square$

**Lemma 19.** *If  $\frac{1}{a} \leq \delta' \leq \frac{c}{a} \leq \theta_1$ , let  $w = X^{\delta'}$ , then*

$$\begin{aligned} \sum_{w \leq p \leq y} w_p \mathcal{S}(\mathcal{A}_p, z) &\leq \xi \pi(X) V(z) \left( a e^{-\gamma} \alpha \int_{\delta'}^{\frac{c}{a}} \left( \frac{1}{s} - u \right) F_2(a(\theta_2 - s)) ds + o(1) \right) \\ &\quad + O \left( q \xi X^{\theta_2 + \varepsilon} + \frac{X^{1+\varepsilon}}{q} \right). \end{aligned}$$

*Proof.* From lemma 18 we obtain

$$\begin{aligned} \sum_{w \leq p \leq y} w_p \mathcal{S}(\mathcal{A}_p, z) &\leq X \xi V_2(z) \left( \sum_{w \leq p \leq y} \frac{w_p}{p} F_2(s'_p) + O \left( (\log X)^{-\frac{1}{6}} \sum_{w \leq p \leq y} \frac{1}{p} \right) \right) \\ &\quad + E_{\mathcal{A}}(X; w, y) \\ &= X \xi V_2(z) \left( \sum_{w \leq p \leq y} \frac{w_p}{p} F_2(s'_p) + o(1) \right) + E_{\mathcal{A}}(X; w, y), \end{aligned}$$

where

$$E_{\mathcal{A}}(X; w, y) := \sum_{w \leq p \leq y} w_p \sum_{d \leq \frac{X^{\theta_2}}{p}} E(X; p, d).$$

Use the same method in Lemma 16 to handle  $\sum_{w \leq p \leq y} \frac{w_p}{p} F_2(s'_p)$  and we obtain that

$$\sum_{w \leq p \leq y} \frac{w_p}{p} F_2(s'_p) = \int_{\delta}^{\frac{1}{u}} \left( \frac{1}{s} - u \right) F_2 \left( \frac{4(\theta_2 - s)}{\alpha} \right) ds + o(1).$$

As for  $E_{\mathcal{A}}(X; w, y)$ , noting that for any  $0 < \theta_2 < 1$ , and for any  $1 \leq B \leq X^{\theta_2}/p$ , we have

$$\sum_{d \sim B} (k, d) = \sum_{c|k} \sum_{\substack{d \sim B \\ c=(k,d)}} c = \sum_{c|k} c \sum_{\substack{d \sim B c^{-1} \\ (d, kc^{-1})=1}} 1 \ll \sum_{c|k} B = B \tau(k),$$

by Abel transformation,

$$\sum_{d \sim B} \frac{(k, d)}{d} \ll \tau(k),$$

which illustrates

$$(6) \quad \sum_{d \leq X^{\theta_2}/p} \frac{(k, d)}{d} \ll \tau(k) \log X.$$

Hence we conclude that

$$\begin{aligned} E_{\mathcal{A}}(X; w, y) &\leq \sum_{w \leq p \leq y} \sum_{d \leq X^{\theta_2}/p} |E(X; p, d)| \\ &\ll \sum_{w \leq p \leq y} \sum_{d \leq X^{\theta_2}/p} \left( q \xi \tau(d) + \frac{X}{pq} \sum_{d=d_1 d_2} \frac{(a, p d_2)(q, d_1)}{d_2 d_1} \right) \\ &\ll q \xi \sum_{w \leq p \leq y} \frac{X^{\theta_2 + \varepsilon}}{p} + \frac{X}{q} \sum_{w \leq p \leq y} \sum_{d_1 d_2 \leq X^{\theta_2}/p} \frac{(a, p d_2)}{p d_2} \cdot \frac{(q, d_1)}{d_1}. \end{aligned}$$

Noticing that (6), the lemma follows immediately.  $\square$

Thus we conclude our results above in a more general form: Given  $z = X^\alpha$ ,  $y = X^\beta$  and  $w = X^{\delta'}$ , where  $\alpha \leq \delta' \leq \beta \leq \theta_1$ , then we have

$$\begin{aligned}
\sum_{z \leq p < y} \mathcal{S}(\mathcal{A}_p, z) &= \sum_{z \leq p < w} \mathcal{S}(\mathcal{A}_p, z) + \sum_{w \leq p < y} \mathcal{S}(\mathcal{A}_p, z) \\
&\leq \sum_{z \leq p < w} \frac{\xi \pi(X) V(z)}{p} F_1(s_p) + \sum_{w \leq p < y} \frac{\xi X V_2(z)}{p} F_2(s'_p) + \text{Error Term} \\
&= \xi \pi(X) V(z) \int_z^w F_1\left(\frac{\log(X^{\theta_1}/t)}{\log z}\right) d \sum_{p \geq t} \frac{1}{p} \\
&\quad + \xi X V_2(z) \int_w^y F_2\left(\frac{\log(X^{\theta_2}/t)}{\log z}\right) d \sum_{p \geq t} \frac{1}{p} + \text{Error Term} \\
&= \xi \pi(X) V(z) \int_z^w \frac{1}{t \log t} F_1\left(\frac{\log(X^{\theta_1}/t)}{\log z}\right) dt \\
&\quad + \xi X V_2(z) \int_w^y \frac{1}{t \log t} F_2\left(\frac{\log(X^{\theta_2}/t)}{\log z}\right) dt + \text{Error Term} \\
&= \xi \pi(X) V(z) \left( \int_\alpha^{\delta'} F_1\left(\frac{\theta_1 - s}{\alpha}\right) \frac{ds}{s} + \frac{e^{-\gamma}}{\alpha} \int_{\delta'}^\beta F_2\left(\frac{\theta_2 - s}{\alpha}\right) \frac{ds}{s} + o(1) \right)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{z \leq p < y} w_p \mathcal{S}(\mathcal{A}_p, z) &\leq \xi \pi(X) V(z) \int_\alpha^{\delta'} \left(\frac{1}{s} - u\right) F_1\left(\frac{\theta_1 - s}{\alpha}\right) ds \\
&\quad + \xi \pi(X) V(z) \left( \frac{e^{-\gamma}}{\alpha} \int_{\delta'}^\beta \left(\frac{1}{s} - u\right) F_2\left(\frac{\theta_2 - s}{\alpha}\right) ds + o(1) \right).
\end{aligned}$$

As shown later in this paper, we can optimize  $\delta'$  to make the upper bounds of  $\sum_{z \leq p < y} \mathcal{S}(\mathcal{A}_p, z)$  or  $\sum_{z \leq p < y} w_p \mathcal{S}(\mathcal{A}_p, z)$  achieve their minimal value, where

$$\delta' = \delta_0 = \theta_2 - \frac{1}{2} \left( A_3 - \sqrt{A_3^2 - 4A_3(\theta_2 - \theta_1)} \right),$$

which is actually very close to  $\theta_1$ . If we take  $a\theta_1 = 6$ , which is a simple but effective choice, then the computations from [10] tell us that

$$b < \frac{\log(1 + e^{24B}) + D - \log 6}{B} - \frac{18}{1 + e^{-24B}} \approx 4.2,$$

while  $a \geq 18$  by Theorem 22. Hence  $\frac{b}{a} < 0.24 < \delta_0$  if  $\rho \geq 15$ , since actually we can take

$$\delta_0 = \frac{2}{3} - \rho - \frac{1}{2} \left( A_3 - \sqrt{A_3^2 - \frac{4A_3}{3}} \right) \pm 10^{-10}.$$

Therefore, we can only use a 2-dimensional sieve to the last term in Lemma 8.

**Lemma 20.** *We have*

$$\mathcal{S}(\mathcal{A}^*, X^{\frac{1}{2}-\eta}) \leq (4\mathfrak{I}(\rho) + o(1)) \frac{\xi \pi(X)}{\log X} + O\left(X^\varepsilon \sum_{r \leq X^\nu} |R_r^*(\beta)|\right),$$

where

$$(7) \quad \sum_{r \leq X^\nu} |R_r^*(\beta)| \ll \xi \pi(X) X^{-\frac{\eta}{2}},$$

with  $\nu = \frac{1-\beta}{2} - \rho - 2\eta$  and  $\mathfrak{I}(\rho)$  is defined by

$$(8) \quad \mathfrak{I}(\rho) := \int_{\frac{\alpha}{4}}^{\frac{1}{4}} \frac{du_1}{u_1(1-u_1-2\rho)} \int_{u_1}^{\frac{1-u_1}{3}} \frac{du_2}{u_2} \int_{u_2}^{\frac{1-u_1-u_2}{2}} \frac{du_3}{u_3(1-u_1-u_2-u_3)}.$$

*Proof.* This follows from [5], Theorem 8.3 and [6].  $\square$

*Remark.* We shall use (7) to give some restrictions in Theorem 22.

*Proof of theorem 14.* We have

$$\sum_{p \geq X^{\frac{1}{a}}} \sum_{h \in \mathcal{A}_{p^2}} 1 \ll \sum_{p \geq X^{\frac{1}{a}}} \frac{\pi(x)\xi}{p^2} \ll \frac{\pi(x)\xi}{X^{\frac{1}{a}}} \ll X^{1-\eta}\xi = o(\xi\pi(X)V(z)).$$

It comes from lemma 15, lemma 16 and lemma 19 that

$$\begin{aligned} \lambda^{-1}\mathcal{W}(\mathcal{A}, u, \lambda) &\geq \lambda^{-1}\mathcal{S}(\mathcal{A}, z) - \mathcal{S}_{\tilde{w}}(\mathcal{A}) + o(\xi\pi(X)V(z)) \\ &= \mathcal{W}_1(\mathcal{A}, u, \lambda) - \mathcal{W}_2(\mathcal{A}, u, \lambda) + o(\xi\pi(X)V(z)), \end{aligned}$$

where

$$\begin{aligned} \mathcal{W}_1(\mathcal{A}, u, \lambda) &:= \lambda^{-1}\mathcal{S}(\mathcal{A}, z) - (c-b) \sum_{X^{\frac{1}{a}} \leq p < X^{\frac{b}{a}}} \mathcal{S}(\mathcal{A}_p, X^{\frac{1}{a}}) \\ &\quad - a \int_{\frac{1}{a}}^{\frac{b+1}{2a}} \left( \sum_{X^s \leq p \leq X^{\frac{b+1}{a}-s}} \mathcal{S}(\mathcal{A}_p, X^s) ds \right) - c \sum_{X^{\frac{b}{a}} \leq p < X^\delta} w_p \mathcal{S}(\mathcal{A}_p, X^{\frac{1}{a}}) \\ &\quad - c \sum_{X^{\frac{1}{a}} \leq p < X^{\frac{b+1}{2a}}} \left( \frac{b+1}{c} - \frac{2u \log p}{\log X} \right) \mathcal{S}(\mathcal{A}_p, p) \\ &\geq \xi\pi(X)V(z) \{ (5c-a)f_1(a\theta_1) - \int_{\frac{1}{a\theta_1}}^{\frac{b+1}{2a\theta_1}} \left( \int_s^{\frac{b+1}{a\theta_1}-s} F_1\left(\frac{1-t}{s}\right) \frac{dt}{t} \right) \frac{ds}{s} \\ &\quad - (c-b) \int_{\frac{1}{a\theta_1}}^{\frac{b}{a\theta_1}} F_1(a\theta_1(1-s)) \frac{ds}{s} - \int_{\frac{b}{a\theta_1}}^\delta \left( \frac{c}{s} - a\theta_1 \right) F_1(a\theta_1(1-s)) ds \\ &\quad - \int_{\frac{1}{a\theta_1}}^{\frac{b+1}{2a\theta_1}} \left( \frac{b+1}{a\theta_1} - 2s \right) F_1\left(\frac{1-s}{s}\right) \frac{ds}{s^2} \} + o(1) \\ &\geq \xi\pi(X)V(z) \{ (5c-a)f_1(\vartheta) - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} \left( \int_s^{\frac{b+1}{\vartheta}-s} F_1\left(\frac{1-t}{s}\right) \frac{dt}{t} \right) \frac{ds}{s} \\ &\quad - (c-b) \int_{\frac{1}{\vartheta}}^{\frac{b}{\vartheta}} F_1(\vartheta(1-s)) \frac{ds}{s} - \int_{\frac{b}{\vartheta}}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_1(\vartheta(1-s)) ds + o(1) \\ &\quad - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} \left( \frac{b+1}{\vartheta} - 2s \right) F_1\left(\frac{1-s}{s}\right) \frac{ds}{s^2} + \int_{\delta}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_1(\vartheta(1-s)) ds \}; \end{aligned}$$



where  $\vartheta = a\theta_1$ ,  $\delta \in [\frac{b}{\vartheta}, \frac{c}{\vartheta}]$ , and

$$\begin{aligned} \mathcal{W}_2(\mathcal{A}, u, \lambda) &:= c \sum_{X^\delta \leq p < X^{\frac{c}{a}}} w_p \mathcal{S}\left(\mathcal{A}_p, X^{\frac{1}{a}}\right) \\ &\leq ae^{-\gamma} \xi \pi(X) V(z) \left( \int_{\delta}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_2(a\theta_2 - \vartheta s) ds + O\left(q \xi X^{\theta_2 + \varepsilon} + \frac{X^{1+\varepsilon}}{q}\right) \right). \end{aligned}$$

To be admissible,  $\theta_2$  can be taken to be any number smaller than  $\frac{2}{3} - \rho$  since  $q \asymp q^{\frac{1}{3} + \rho + \eta}$ . Take  $\theta_2 = \frac{2}{3} - \rho - \varepsilon$  and  $\frac{\theta_2}{\theta_1} \theta \rightarrow \theta := \frac{\frac{2}{3} - \rho}{\frac{2}{3} - \rho}$  as  $\varepsilon \rightarrow 0^+$ , thus by continuity and Lemma 20 we have, when  $\varepsilon$  is sufficiently small,

$$\mathcal{J}(\lambda) \geq \frac{ae^{-\gamma} \lambda \xi \pi(X)}{\log X} \mathcal{H}(\vartheta, b, c),$$

where

$$\begin{aligned} \mathcal{H}(\vartheta, b, c) &= \mathcal{H}_\delta(\vartheta, b, c) := (5c - a)f_1(\vartheta) - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} \left( \int_s^{\frac{b+1}{\vartheta} - s} F_1\left(\frac{1-t}{s}\right) \frac{dt}{t} \right) \frac{ds}{s} \\ &\quad - (c - b) \int_{\frac{1}{\vartheta}}^{\frac{b}{\vartheta}} F_1(\vartheta(1-s)) \frac{ds}{s} - \int_{\frac{b}{\vartheta}}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_1(\vartheta(1-s)) ds + o(1) \\ &\quad - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} \left( \frac{b+1}{\vartheta} - 2s \right) F_1\left(\frac{1-s}{s}\right) \frac{ds}{s^2} + \int_{\delta}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_1(\vartheta(1-s)) ds \\ &\quad - ae^{-\gamma} \left( \int_{\delta}^{\frac{c}{\vartheta}} \left( \frac{c}{s} - \vartheta \right) F_2(a\theta_2 - \vartheta s) ds \right) - \frac{4e^{\gamma}c}{a} \mathfrak{J}(\rho). \end{aligned}$$

Then

$$\mathcal{H}'_c(\vartheta, b, c) = 2f_1(\vartheta) - \int_{\frac{1}{\vartheta}}^{\delta} F_1(\vartheta(1-s)) \frac{ds}{s} - ae^{-\gamma} \int_{\delta}^{\frac{c}{\vartheta}} F_2(\vartheta(\theta - s)) \frac{ds}{s} - \frac{4e^{\gamma}}{a} \mathfrak{J}(\rho),$$

and

$$\begin{aligned} \mathcal{H}'_b(\vartheta, b, c) &= -f_1(\vartheta) + \int_{\frac{1}{\vartheta}}^{\frac{b}{\vartheta}} F_1(\vartheta(1-s)) \frac{ds}{s} - \frac{1}{\vartheta} \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} F_1\left(\frac{1-s}{s}\right) \frac{ds}{s^2} \\ &\quad - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} F_1\left(\frac{\vartheta s + \vartheta - 1 - b}{\vartheta s}\right) \frac{ds}{s(b+1-\vartheta s)}. \end{aligned}$$

Assume that  $\vartheta \geq 4$  and  $b \geq \vartheta - 3$ , then we have

$$\begin{aligned} \mathcal{H}''_{bc}(\vartheta, b, c) &= 0; \\ \mathcal{H}''_{b^2}(\vartheta, b, c) &= \frac{1}{b} F_1(\vartheta - b) - \left( -\frac{1}{2\vartheta - 1 - b} + \frac{1}{\vartheta - b} - \frac{1}{b+1} + \frac{1}{b} \right) \\ &\quad - \frac{2}{(b+1)^2} F_1\left(\frac{2\vartheta}{b+1} - 1\right) = 0; \\ \mathcal{H}''_{c^2}(\vartheta, b, c) &= -\frac{ae^{-\gamma} F_2(\vartheta\theta - c)}{c}. \end{aligned}$$

Thus we can write  $\mathcal{H}(\vartheta, b, c)$  as

$$\mathcal{H}(\vartheta, b, c) = 2e^{\gamma} (A(\vartheta)b + B(\vartheta)c + D(\vartheta) + \mathfrak{F}(\vartheta, c)).$$

where

$$\mathfrak{F}(\vartheta, \vartheta\theta) = 0 \quad \text{and} \quad \mathfrak{F}'(\vartheta, c) = -\frac{a}{2e^{2\gamma}} \int_{\delta}^{\frac{c}{\vartheta}} F_2(\vartheta(\theta - s)) \frac{ds}{s},$$

and  $A(\vartheta)$ ,  $B(\vartheta)$ ,  $D(\vartheta)$  are determined by

$$\begin{aligned}
2e^\gamma (A(\vartheta) + B(\vartheta)) &= f_1(\vartheta) - \frac{1}{\vartheta} \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} F_1 \left( \frac{1-s}{s} \right) \frac{ds}{s^2} - \frac{4e^\gamma}{a} \mathfrak{I}(\rho) \\
&\quad - \int_{\frac{1}{\vartheta}}^{\frac{b+1}{2\vartheta}} F_1 \left( \frac{\vartheta s + \vartheta - 1 - b}{\vartheta s} \right) \frac{ds}{s(b+1-\vartheta s)} \\
&= f_1(\vartheta) - \frac{2e^\gamma}{\vartheta} \log \frac{\delta(2\vartheta - \vartheta\delta - 1)}{(1-\delta)(\vartheta\delta + 1)} - f_1(\vartheta) + \frac{2}{\vartheta\delta + 1} f_1 \left( \frac{2\vartheta}{\vartheta\delta + 1} \right) - \frac{4e^\gamma}{a} \mathfrak{I}(\rho) \\
&= \frac{2e^\gamma}{\vartheta} \log \frac{1-\delta}{\delta} - \frac{4e^\gamma}{a} \mathfrak{I}(\rho);
\end{aligned}$$

Also we have, by direct computation,

$$B(\vartheta) = e^{-\gamma} f_1(\vartheta) - \frac{1}{2e^\gamma} \int_{\frac{1}{\vartheta}}^{\delta} F_1(\vartheta(1-s)) \frac{ds}{s} - \frac{2}{a} \mathfrak{I}(\rho),$$

so

$$\begin{aligned}
A(\vartheta) &= -e^{-\gamma} f_1(\vartheta) + \frac{1}{2e^\gamma} \int_{\frac{1}{\vartheta}}^{\delta} F_1(\vartheta(1-s)) \frac{ds}{s} + \frac{1}{\vartheta} \log \frac{1-\delta}{\delta}; \\
D(\vartheta) &= \frac{1}{2e^\gamma} \mathcal{H}(\vartheta, \vartheta\theta, \vartheta\theta) - \delta \log \frac{1-\delta}{\delta} + \frac{2\vartheta\theta}{a} \mathfrak{I}(\rho).
\end{aligned}$$

Thus, then by the continuity of  $F_2$  we obtain theorem 14.  $\square$

## 6. PROOF OF THEOREM 1

It is obvious that by Corollary 4 and Theorem 14 we have:

**Theorem 21.** *The restriction from the main terms is given by*

$$\begin{cases} 1 \leq b \leq c \leq a = \frac{\vartheta}{\theta_1} \\ b = 1 \text{ or } a \geq 3c + b + 1, \text{ if } b \geq 3 \\ \frac{b}{\vartheta} \leq \delta_0 \leq \frac{c}{\vartheta} \\ \max_{\frac{b}{\vartheta} \leq \delta \leq \frac{c}{\vartheta}} \mathcal{H}_\delta(\vartheta, b, c) > 0. \end{cases}$$

where  $\mathcal{H}_{\delta_0}(\vartheta, b, c)$  is defined by (5) with  $F_2$  defined as before Theorem 14.

**Theorem 22.** *The restrictions from the error terms are given as the following inequation systems:*

$$\begin{cases} 0 < \rho < \min \left\{ \frac{1}{6}, \frac{1}{a} \right\}, \\ \theta_1 + \rho < \frac{1}{3}, \theta_1 > 0. \end{cases}$$

*Proof.* In Corollary 13 and Lemma 12 above, where we show that

$$\sum_{r \leq X^\alpha} \frac{1}{r} \max_{N \leq X} \sum_{l=1}^{rY} \left| \sum_{n \leq N} \Lambda(n) e \left( \frac{\alpha n l}{r q} \right) \right| \ll \pi(X) X^{-\eta},$$

with  $Y \asymp X^{\rho+\eta}$ , we have to make sure that all the parameters satisfy the assumptions of those lemmas.

Divide the intervals into dyadic segments and thus we have the following estimation:

$$\begin{aligned} \sum_i \sum_{i_j} X^\varepsilon \left( X^{\frac{2}{3}} T_i + X^{\frac{11}{12}} \left( \frac{T_i}{R_i} \right)^{\frac{1}{2}} \right) &\ll X^{2\varepsilon} X^{\frac{2}{3} + \theta_1 + \rho + \eta} + X^{\frac{11}{12}} \sum_i \sum_{i_j} X^{\frac{\rho + \eta}{2}} \\ &\ll X^{2\varepsilon} X^{\frac{2}{3} + \theta_1 + \rho + \eta} + X^{\frac{11}{12} + \varepsilon + \frac{\rho + \eta}{2}}, \end{aligned}$$

where  $L_{i_j} \leq R_i Y \ll R_i X^{\rho + \eta}$ ,  $T_i \ll R_i X^{\rho + \eta}$  and for simplicity we omit the precise range of  $i$  and  $j$ , actually, only the bound  $i, j \ll \log X$  matters.

Therefore, we get our restrictions as below:

$$\begin{cases} 2\varepsilon + \frac{2}{3} + \theta_1 + \rho + \eta < 1 - \eta, \\ \frac{11}{12} + \varepsilon + \frac{\rho + \eta}{2} < 1 - \eta, \end{cases}$$

i.e.

$$\begin{cases} \theta_1 + \rho < \frac{1}{3}, \\ \rho < \frac{1}{6}. \end{cases}$$

Now let's consider another estimation from (7). By assumption, we have  $X^{\rho + \beta + \eta} < X^{\rho + \frac{1}{4} + \eta} < q < X^{\frac{3}{4} - \eta} < X^{1 - \beta - \eta}$ . Additionally, by Lemma 12, there should be

$$\xi X^{1 + 3\varepsilon} \sum_i \sum_{i_j} \left( L_{i_j} + \frac{R_i}{X^\beta} \right) \left( X^{\frac{\beta - 1}{2}} + X^{-\beta} R_i^{-\frac{1}{2}} \left( L_{i_j} + \frac{R_i}{X^\beta} \right)^{-\frac{1}{2}} \right) \ll \xi \pi(X) X^{-\eta}.$$

While

$$\begin{aligned} \xi X^{\frac{\beta + 1}{2} + 3\varepsilon} \sum_i \sum_{i_j} \left( L_{i_j} + \frac{R_i}{X^\beta} \right) \\ \ll \xi X^{\frac{\beta + 1}{2} + 4\varepsilon} \left( X^{\frac{1 - \beta}{2} - \rho - 2\eta} + X^{-\beta} X^{\frac{1 - \beta}{2} - 2\eta} \right) \ll \xi X^{1 - \rho - 2\eta + 4\varepsilon} + \xi X^{1 - \beta - 2\eta}, \end{aligned}$$

and

$$\begin{aligned} \xi X^{1 + 3\varepsilon} \sum_i \sum_{i_j} \left( X^{-\frac{\beta}{2}} R_i^{-\frac{1}{2}} \left( L_{i_j} + \frac{R_i}{X^\beta} \right)^{\frac{1}{2}} \right) \\ \ll \xi X^{1 + 3\varepsilon} \sum_i \sum_{i_j} \left( X^{-\frac{\beta}{2}} R_i^{-\frac{1}{2}} L_{i_j}^{\frac{1}{2}} + X^{-\beta} \right) \ll \xi X^{1 + 4\varepsilon + \frac{\rho + \eta}{2} - \frac{\beta}{2}}, \end{aligned}$$

so it suffices to have the restriction:  $1 + 4\varepsilon + \frac{\rho + \eta}{2} - \frac{\beta}{2} < 1 - \frac{\eta}{2} - \varepsilon$ , which could be deduced by the condition:  $\beta > \rho \Leftarrow \frac{1}{a} > \rho$ . This completes the proof.  $\square$

Combine all the restrictions from Theorem 21 and Theorem 22. Take  $\theta_1 = \frac{1}{3} - \rho - 10^{-12}$ , then insert this into the above conditions, with the help of the software *Mathematica 9*, we find that  $\rho = \frac{1}{118}$  satisfies the restrictions above, when  $b = 1$ ,  $c = 3.98$ ,  $\vartheta = 4.07$  (hence  $a \approx 12.5285$ ), noting that  $a$  slight larger than  $\frac{4}{\alpha}$  in [6]. Thus we have proven that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{118}}.$$

*Remark.* We thus see that in our situation Laborde's weight is not better than Richert's weight because of the effect from  $\mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2} - \eta})$ , since  $\frac{\mathcal{I}(\rho)}{a} = a\mathcal{I}(\rho, a)$  grows faster than  $f_1(\vartheta)$  when  $\vartheta \geq 4$ . When  $b > 1$ , which forces that  $a \geq 3c + b + 1 > 5$ , the contribution of  $\mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2} - \eta})$  is just too large for our purpose. If we just take  $\delta = \alpha$  as Harman did in [6], then by optimizing the parameters directly we have  $\tau < \frac{1}{146}$  and we can take  $\tau = \frac{1}{147}$ .

## 7. ESTIMATES FOR EXPONENTIAL SUMS II

**Lemma 23** ([1]). *For any  $\iota \in [0, 1]$ , let*

$$f_{h,\iota,\varsigma}(x) := h(x + \iota)^\gamma + \varsigma x,$$

*where  $h \in \mathbb{N}$  and  $\varsigma$  is an arbitrary constant. Take  $\sigma$  satisfying the restriction  $\sigma < \frac{9\gamma-8}{12}$ . Then any sufficiently small  $\eta > 0$ , we have*

$$\min \left\{ 1, \frac{X^{1-\gamma}}{H} \right\} \sum_{h \sim H} \left| \sum_{n \sim X} \Lambda(n) e(f_{h,\iota,\varsigma}(n)) \right| \ll_\eta X^{1-\sigma-3\eta},$$

*where  $H \leq X^{1-\gamma+\sigma+\varepsilon}$ .*

*Remark.* We should point out that the  $O$ -constant is independent of  $\iota$  and  $\varsigma$ , namely, it's uniform for  $\varsigma$ , because only the behavior of  $f''_{h,\varsigma}(x)$  is used when handling sums of booth Type I and Type II, after using Heath-Brown's identity (see [7]). This is a critical property as we will see in our situation we actually need to bound a mean estimate of the form

$$\sum_{d \sim D} \sum_{l \sim L} |b_l| \sum_{h \sim H} \frac{1}{h} \max_{N \leq X} \left| \sum_{n \sim N} \Lambda(n) e \left( h(n + \iota)^\gamma + \frac{aln}{qd} \right) \right|.$$

We are showing the level of distribution is  $\theta_3 = \frac{9\gamma-8}{12} - \rho$ .

In this section we aim to prove the following lemma:

**Lemma 24.** *For  $c \in (1, \frac{755}{662})$ ,  $\theta_3 = \frac{1+9(\gamma-1)}{12} - \rho$ , we have*

$$(9) \quad \sum_{d \leq X^{\theta_3}} \frac{\xi}{d} \max_{N \leq X} \sum_{l=1}^{dY} \left| \sum_{\substack{n \leq N \\ n \in \mathcal{P}}} \Lambda(n) e \left( \frac{anl}{dq} \right) \right| \ll \xi \pi_c(X^\gamma) X^{-\eta},$$

*where  $\gamma = \frac{1}{c}$  and  $\mathcal{P} := \{ \lfloor n^c \rfloor : n \in \mathbb{N} \}$ .*

*Proof.* It is clearly that  $p = \lfloor n^c \rfloor$  if and only if there exists a nonnegative  $\nu < 1$  such that  $n^c = p + \nu$ , which, by a direct check, is equivalent to

$$\lfloor -p^\gamma \rfloor - \lfloor -(p+1)^\gamma \rfloor = 1,$$

where  $\gamma$  is taken to be the inverse of  $c$  traditionally.

Hence we can take  $\phi(n) := \lfloor -n^\gamma \rfloor - \lfloor -(n+1)^\gamma \rfloor$  to be a characteristic function of  $\mathcal{P}$ , and thus for any  $N \leq X$ , we have

$$\sum_{l=1}^{dY} \left| \sum_{\substack{n \leq N \\ n \in \mathcal{P}}} \Lambda(n) e \left( \frac{anl}{dq} \right) \right| = \sum_{l=1}^{dY} \left| \sum_{n \leq N} \phi(n) \Lambda(n) e \left( \frac{anl}{dq} \right) \right| \leq \mathcal{E}_1(N, d) + \mathcal{E}_2(N, d),$$

where

$$(10) \quad \mathcal{E}_1(N, d) := \sum_{l=1}^{dY} \left| \sum_{n \leq N} ((n+1)^\gamma - n^\gamma) \Lambda(n) e \left( \frac{anl}{dq} \right) \right|,$$

and

$$\mathcal{E}_2(N, d) := \sum_{l=1}^{dY} \left| \sum_{n \leq N} (\{-n^\gamma\} - \{-(n+1)^\gamma\}) \Lambda(n) e \left( \frac{anl}{dq} \right) \right|.$$

We will see later that  $\mathcal{E}_1(N, d)$  and  $\mathcal{E}_2(N, d)$  are different types of exponential sums, and the former is algebraic, while the latter is analytic. Hence we use different methods to handle them respectively.

**Estimate of  $\mathcal{E}_1(N, d)$ :** Write  $\mathcal{E}_1(N, d)$  in an integral form and integral by parts we have

$$\begin{aligned} \mathcal{E}_1(N, d) &:= \sum_{l=1}^{dY} \left| \int_1^N ((t+1)^\gamma - t^\gamma) d \sum_{n \leq t} \Lambda(n) e\left(\frac{anl}{dq}\right) \right| \\ &= \int_1^N ((t+1)^\gamma - t^\gamma) d \left( \sum_{l=1}^{dY} c_l \sum_{n \leq t} \Lambda(n) e\left(\frac{anl}{dq}\right) \right) \\ &\leq \int_1^N \max_{T \leq N} \left| \sum_{l=1}^{dY} c_l \sum_{n \leq t} \Lambda(n) e\left(\frac{anl}{dq}\right) \right| \left( (t+1)^{\gamma-1} - t^{\gamma-1} + O\left(\frac{1}{N}\right) \right) dt \\ &\ll \max_{T \leq N} \sum_{l=1}^{dY} \left| \sum_{n \leq T} \Lambda(n) e\left(\frac{anl}{dq}\right) \right|, \end{aligned}$$

where  $c_l = e^{i\theta_l}$ , here  $\theta_l$  is the principle argument of the inner sum in (10). Thus by Lemma 12 we have  $YX^{\frac{2}{3}+\theta_3} \ll \pi_c(X^\gamma)X^{-\eta}$ , deducing that

$$\theta_3 \leq \gamma - \frac{2}{3} - \rho - \varepsilon.$$

**Estimate of  $\mathcal{E}_2(N, d)$ :** Take  $\eta = 3\varepsilon$ . By Lemma 7 we have

$$\begin{aligned} \mathcal{E}_2(N, d) &= \sum_{l=1}^{dY} c_l \sum_{n \leq N} \left( \sum_{1 \leq |h| \leq X^{1-\gamma+\sigma+\varepsilon}} \frac{e(h(n+1)^\gamma - e(hn^\gamma))}{2\pi i h} \right) \Lambda(n) e\left(\frac{anl}{dq}\right) \\ &\quad + O\left(X^{\gamma-1-\sigma} \sum_{l=1}^{dY} \sum_{n \leq N} \Lambda(n)\right) \\ &= \sum_{l=1}^{dY} c_l \sum_{1 \leq |h| \leq X^{1-\gamma+\sigma+\varepsilon}} \sum_{n \leq N} \Lambda(n) \frac{\mathcal{E}_2^1(N, d) - \mathcal{E}_2^0(N, d)}{2\pi i h} + O(dYX^{\gamma-\sigma+\eta}). \end{aligned}$$

where

$$\mathcal{E}_2^\iota(N, d) := \sum_{l=1}^{dY} c_l \sum_{1 \leq |h| \leq X^{2+\varepsilon}} \frac{1}{h} \sum_{n \leq N} \Lambda(n) e\left(h(n+\iota)^\gamma + \frac{anl}{dq}\right),$$

for  $\iota \in \{0, 1\}$ . We split the summation range into dyadic segments, a typical one is

$$\begin{aligned} \mathcal{E}_{2_j}^\iota(N, d) &:= \sum_{l=1}^{dY} c_l \sum_{h \sim H} \frac{1}{h} \sum_{n \leq N} \Lambda(n) e\left(h(n+\iota)^\gamma + \frac{anl}{dq}\right) \\ &\ll \sum_{l=1}^{dY} \frac{1}{H} \sum_{h \sim H} \left| \sum_{n \leq N} \Lambda(n) e\left(h(n+\iota)^\gamma + \frac{anl}{dq}\right) \right|, \end{aligned}$$

where  $H$  is of the form  $2^j X^{1-\gamma}$ , and  $j \ll \log X$  since  $H \leq X^{1-\gamma+\sigma+\varepsilon}$ . Hence by Lemma 23 we have

$$\begin{aligned} \sum_{d \leq X^{\theta_3}} \frac{\xi}{d} \max_{N \leq X} |\mathcal{E}_2(N, d)| &\ll \sum_{\iota \in \{0, 1\}} \sum_{j \ll \log X} \sum_{d \leq X^{\theta_3}} \frac{\xi}{d} \max_{N \leq X} |\mathcal{E}_{2_j}^\iota(N, d)| \\ &\ll \sum_{d \leq X^{\theta_3}} \xi Y X^{\gamma-\sigma-2\eta} \ll \xi^2 \pi_c(X^\gamma) X^{\theta_3-\sigma-\eta}. \end{aligned}$$

So it suffices to take  $\theta_3 \leq \frac{1+9(\gamma-1)}{12} - \rho$ .

Combining the above discussion we thus obtain (9).  $\square$

## 8. PROOF OF THEOREM 2

Denote by

$$\hat{\mathcal{B}} := \left\{ \left\lfloor \frac{b+pa}{q} \right\rfloor : p \leq X, p \in \mathcal{P}, \left\| \frac{b+pa}{q} \right\| < \frac{\xi}{2} \right\},$$

where  $\mathcal{P} := \{ \lfloor n^c \rfloor : n \in \mathbb{N} \}$ . By taking  $M \asymp \frac{dX^\eta}{\xi}$  in Lemma 6 we have for and  $d \in \mathbb{N}$

$$\begin{aligned} \#\mathcal{B}_d &= \sum_{\substack{p \leq X, p \in \mathcal{P} \\ d \mid \left\lfloor \frac{ap+b}{q} \right\rfloor \\ \left\| \frac{ap+b}{q} \right\| < \frac{\xi}{2}}} 1 = \sum_{\substack{p \leq X, p \in \mathcal{P} \\ \left\| \frac{ap+b}{dq} \right\| < \frac{\xi}{2d}}} 1 = \sum_{p \leq X, p \in \mathcal{P}} \chi \left( \frac{ap+b}{dq} \right) \\ &= \frac{\pi_c(X^\gamma)\xi}{d} + E(\mathcal{B}_d) + O \left( \frac{\xi \pi_c(X^\gamma)X^{-\eta}}{d} \right), \end{aligned}$$

where

$$\sum_{p \leq X, p \in \mathcal{P}} \sum_{1 \leq |l| \leq M} a_l e \left( \frac{(ap+b)l}{qd} \right) \leq E(\mathcal{B}_d) \leq \sum_{p \leq X, p \in \mathcal{P}} \sum_{1 \leq |l| \leq M} b_l e \left( \frac{(ap+b)l}{qd} \right)$$

with  $|a_l| + |b_l| \ll \frac{\xi}{d}, \forall 1 \leq |l| \leq M$ .

As shown in Lemma 15, by partial summation we have

$$\begin{aligned} E(\mathcal{B}_d) &\ll \max_{N \leq X} \frac{1}{\log X} \sum_{1 \leq |l| \leq M} (|a_l| + |b_l|) \left| \sum_{n \leq N, p \in \mathcal{P}} \Lambda(n) e \left( \frac{anl}{qd} \right) \right| \\ &\ll \max_{N \leq X} \frac{\xi}{d} \sum_{l=1}^{dY} \left| \sum_{n \leq N, p \in \mathcal{P}} \Lambda(n) e \left( \frac{anl}{qd} \right) \right|, \end{aligned}$$

so the density function of sequence  $\mathcal{B}$  is  $g_3(d) = \frac{1}{d}$ , and the corresponding level of distribution  $\theta_3$  can be taken to be  $\frac{1+9(\gamma-1)}{12} - \rho$ .

Since the level here is quite small, there might be little room for other sieve techniques. Thus we choose to use Laborde's results to deal with  $\mathcal{B}$  directly.

**Lemma 25.** *There are infinitely many  $P_r$  in  $\mathcal{B}$  if*

$$\frac{c}{\theta_3} \leq r - 0.144.$$

*Proof.* This is essentially Theorem 3 of [10]. However, the upper bound for  $\Lambda$  there can actually be taken to be 0.144, since

$$\frac{\log 6 - B - D}{6B} - \frac{\log(1 + e^{-78B})}{6B} \approx 0.144002.$$

So we can take 0.144 rather than 0.145 in the statement of Laborde's theorem. This leads us to take  $\rho = \frac{1}{180}$ , otherwise, we can only take  $\rho = \frac{1}{181}$ .  $\square$

Take  $r = 13$  and Theorem 2 follows immediately.

*Remark.* Similarly, we can also use 2-dimensional sieve to sharp the range of  $\rho$ .

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